

On Dynamics in Selfish Network Creation

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Abstract

We consider the dynamic behavior of several variants of the Network Creation Game, introduced by Fabrikant et al. [PODC'03]. For this, we treat these games as sequential-move games and analyze whether selfish play eventually leads to an equilibrium state of the game. We show that fast convergence is guaranteed for all versions of Swap Games, if the initial network is a tree. Furthermore, we show that this process can be sped up to an almost optimal number of moves by employing a very natural move policy. Unfortunately, these positive results are no longer true if the initial network has cycles and we show that even one non-tree edge suffices to destroy the convergence-guarantee. Moreover, we show that on non-tree networks *no* move policy can enforce convergence. We extend our negative results to the well-studied original version, where agents are allowed to buy and delete edges. That is, we show that in the original model there is no convergence guarantee – even if all agents play optimally.

1 Introduction

The Internet is arguably one of the most important artificially created networks of our time. Understanding Internet-like networks and their implications on our life is a recent endeavor undertaken by researchers from different research communities and seems a very important and challenging task. Such networks are difficult to analyze since they are created by a multitude of selfish entities (e.g. Internet Service Providers) which modify the infrastructure of parts of the network (e.g. their Autonomous Systems) to improve their service quality. The classical field of Game Theory provides the tools for analyzing such decentralized processes and from this perspective the Internet can be seen as an equilibrium state of an underlying game played by selfish agents.

Within the last decade several such games have been proposed and analyzed. We will focus on the line of works which consider Network Creation Games, as introduced by Fabrikant et al. [11]. These games are very simple but they contain an interesting trade-off between an agent's investment in infrastructure and her obtained usage quality. Agents aim to invest as little as possible but at the same time they want to achieve a good connection to all other agents in the network. Network Creation Games and several variants have been studied intensively, but, to the best of our knowledge, almost all these works exclusively focus on properties of the equilibrium states of the game. With this focus, the game is usually considered to be a one-shot simultaneous-move game. However, the Internet was not created

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in “one shot”. It has rather evolved from an initial network, the ARPANET, into its current shape by repeated infrastructural changes performed by selfish agents who entered or left the stage at some time in the process. For this reason, we focus on a more dynamic point of view: We analyze the properties of the network creation *processes* induced by the sequential-move version of the known models of selfish network creation.

1.1 Models and Definitions

We consider several versions of a network creation process performed by n selfish agents. In all versions we consider networks, where every node corresponds to an agent and undirected links connect network nodes. The creation process is based on an underlying Network Creation Game (NCG) and can be understood as a dynamic process where agents sequentially perform strategy-changes in the NCG. In such games, the strategies of the agents determine which links are present in the network and any strategy-profile, which is a vector of the strategies of all n agents, determines the induced network. But this also works the other way round: Given some network $G = (V, E, o)$, where V is the set of n vertices, E is the set of edges and $o : E \rightarrow V$ is the *ownership-function*, which assigns the ownership of an edge to one of its endpoints, then G completely determines the current strategies of all n agents of the NCG. Starting from a network G_0 , any sequence of strategy-changes by agents can thus be seen as a sequence of networks G_0, G_1, G_2, \dots , where the network G_{i+1} arises from the network G_i by the strategy-change of exactly one agent. In the following, we will write xy or yx for the undirected edge $\{x, y\} \in E$. In figures we will indicate edge-ownership by directing edges away from their owner.

The creation process starts in an initial state G_0 , which we call the *initial network*. A step from state G_i to state G_{i+1} consists of a *move* by one agent. A move of agent u in state G_i is the replacement of agent u ’s pure strategy in G_i by another *admissible* pure strategy of agent u . The induced network after this strategy-change by agent u then corresponds to the state G_{i+1} . We consider only improving moves, that is, strategy-changes which strictly decrease the moving agent’s cost. The cost of an agent in G_i depends on the structure of G_i and it will be defined formally below. If agent u in state G_i has an admissible new strategy which yields a strict cost decrease for her, then we call agent u *unhappy in network G_i* and we let U_i denote the set of all unhappy agents in state G_i . Only one agent can actually move in a state of the process and this agent $u \in U_i$, whose move transforms G_i into G_{i+1} , is called *the moving agent in state G_i* . In any state of the process the identity of the moving agent is determined by the *move policy* of the process, which we will define below. The process stops in some state G_j if no agent wants to perform a move, that is, if $U_j = \emptyset$, and we call the resulting networks *stable*. Clearly, stable networks correspond to pure Nash equilibria of the underlying NCG.

Depending on what strategies are admissible for an agent in the current state, there are several variants of this process, which we call *game types*:

- In the *Swap Game* (SG), introduced as “Basic Network Creation Game” by Alon et al. [2], the strategy S_u of an agent u in the network G_i is the set of neighbors of vertex u in G_i . The new strategy S_u^* is admissible for agent u in state G_i , if $|S_u| = |S_u^*|$ and $|S_u \cap S_u^*| = |S_u| - 1$. Intuitively, admissible strategies in the SG are strategies

which replace one neighbor x of u by another vertex y . Note, that this corresponds to “swapping” the edge ux from x towards y , which is the replacement of edge ux by edge uy . Furthermore, observe, that in any state both endpoints of an edge are allowed to swap this edge. Technically, this means that the ownership of an edge has no influence on the agents’ strategies.

- The *Asymmetric Swap Game* (ASG), recently introduced by Mihalák and Schlegel [14], is similar to the SG, but here the ownership of an edge plays a crucial role. Only the owner of an edge is allowed to swap the edge in any state of the process. The strategy S_u of agent u in state G_i is the set of neighbors in G_i to which u owns an edge and the strategy S_u^* is admissible for agent u in state G_i , if $|S_u| = |S_u^*|$ and $|S_u \cap S_u^*| = |S_u| - 1$. Hence, in the ASG the moving agents are allowed to swap one own edge.
- In the *Greedy Buy Game* (GBG), recently introduced by Lenzner [13], agents have more freedom to act. In any state, an agent is allowed to buy or to delete or to swap one own edge. Hence, the GBG can be seen as an extension of the ASG. The strategy S_u of agent u in state G_i is defined as in the ASG, but the set of admissible strategies is larger: S_u^* is admissible for agent u in state G_i if (1) $|S_u^*| = |S_u| + 1$ and $S_u \subset S_u^*$ or (2) if $|S_u^*| = |S_u| - 1$ and $S_u^* \subset S_u$ or (3) if $|S_u| = |S_u^*|$ and $|S_u \cap S_u^*| = |S_u| - 1$.
- The *Buy Game* (BG), which is the original version of an NCG and which was introduced by Fabrikant et al. [11], is the most general version. Here agents can perform arbitrary strategy-changes, that is, agents are allowed to perform any combination of buying, deleting and swapping of own edges. The strategy S_u of agent u in G_i is defined as in the ASG, but an admissible strategy for agent u is any set $S_u^* \subseteq V \setminus \{u\}$.

The *cost* of an agent u in network G_i has the form $c_{G_i}(u) = e_{G_i}(u) + \delta_{G_i}(u)$, where $e_{G_i}(u)$ denotes the *edge-cost* and $\delta_{G_i}(u)$ denotes the *distance-cost* of agent u in the network G_i . If edges have owners, then we assume that each edge has cost $\alpha > 0$, which is a fixed constant, and this cost has to be paid fully by the owner. Hence, if agent u owns k edges in state G_i , then $e_{G_i}(u) = \alpha k$. In the SG, where edges do not have owners, we simply omit the edge-cost term in the cost function. There are two variants of distance-cost functions. In the SUM-version, we have $\delta_{G_i}(u) = \sum_{v \in V(G_i)} d_{G_i}(u, v)$, if the network G_i is connected and $\delta_{G_i}(u) = \infty$, otherwise. In the MAX-version, we have $\delta_{G_i}(u) = \max_{v \in V(G_i)} d_{G_i}(u, v)$, if G_i is connected and $\delta_{G_i}(u) = \infty$, otherwise. In both cases $d_{G_i}(u, v)$ denotes the shortest path distance between vertex u and v in the undirected graph G_i .

The move policy specifies for any state of the process, which of the unhappy agents is allowed to perform a move. From a mechanism design perspective, the move policy is a way to enforce coordination and to guide the process towards a stable state. We will focus on the *max cost policy*, where the agent having the highest cost is allowed to move and ties among such agents are broken arbitrarily. Sometimes we will assume that an adversary chooses the worst possible moving agent. Note, that the move policy only specifies who is allowed to move, not which specific move has to be performed. We do not consider such strong policies since we do not want to restrict the agents’ freedom to act.

Any combination of the four game types, the two distance functions and some move policy together with an initial network completely specifies a network creation process. We

will abbreviate the names, e.g. by calling the Buy Game with the SUM-version of the distance cost the SUM-BG.

A cyclic sequence of networks C_1, \dots, C_j , where network $C_{i+1 \bmod j}$ arises from network $C_{i \bmod j}$ by an improving move of one agent is called a *better response cycle*. If every move in such a cycle is a *best response move*, which is a strategy-change towards an admissible strategy which yields the largest cost decrease for the moving agent, then we call such a cycle a *best response cycle*. Clearly, a best response cycle is a better response cycle, but the existence of a better response cycle does not imply the existence of a best response cycle.

1.2 Classifying Games According to their Dynamics

Analyzing the convergence processes of games is a very rich and diverse research area. We will briefly introduce two well-known classes of finite strategic games: *games having the finite improvement property* (FIPG) [16] and *weakly acyclic games* (WAG) [17].

FIPG have the most desirable form of dynamic behavior: Starting from any initial state, every sequence of improving moves must eventually converge to an equilibrium state of the game, that is, such a sequence must have finite length. It was shown in [16] that a finite game is a FIPG if and only if there exists a *generalized ordinal potential function* Φ , which maps strategy-profiles to real numbers and has the property that if the moving agent's cost decreases, then the potential function value decreases as well. Stated in our terminology, this means that $\Phi : \mathcal{G}_n \rightarrow \mathbb{R}$, where \mathcal{G}_n is the set of all networks on n nodes, and we have

$$c_{G_i}(u) - c_{G_{i+1}}(u) > 0 \Rightarrow \Phi(G_i) - \Phi(G_{i+1}) > 0,$$

if agent u is the moving agent in the network G_i . Clearly, no FIPG can admit a better response cycle.

An especially nice subclass of FIPG are games that are guaranteed to converge to a stable state in a number of steps which is a polynom of the size of the game. We call this subclass poly-FIPG.

Weakly acyclic games are a super-class of FIPG. Here it is not necessarily true that *any* sequence of improving moves must converge to an equilibrium but we have that from any initial state there exists *some* sequence of improving moves which enforces convergence. A subclass of WAG are games where from any initial state there exists a sequence of best response moves, which leads to an equilibrium. We call those games *weakly acyclic under best response*, BR-WAG for short.

The above mentioned classes of finite strategic games are related as follows:

$$\text{poly-FIPG} \subset \text{FIPG} \subset \text{BR-WAG} \subset \text{WAG}.$$

The story does not end here. Very recently, Apt and Simon [3] have classified WAG in much more detail by introducing a “scheduler”, which is a moderating super-player who guides the agents towards an equilibrium. Moreover, Fabrikant et al. [10] showed that the existence of a pure Nash equilibrium in every subgame implies weak acyclicity.

1.3 Related Work

The original model of Network Creation Games, which we call the SUM-BG, was introduced a decade ago by Fabrikant et al. [11]. Their motivation was to understand the creation of

Internet-like networks by selfish agents without central coordination. In the following years, several variants were proposed: The MAX-BG [8], the SUM-SG and the MAX-SG [2], the SUM-ASG and the MAX-ASG [14], the SUM-GBG and the MAX-GBG [13], a bounded budget version [9] and an edge-restricted version [4, 7]. All these works, as well as several others, e.g. [1] and [15], focus exclusively on properties of stable networks or on the complexity of computing an agent’s best response. To the best of our knowledge, the dynamic behavior of most of these variants, including best response dynamics in the well-studied original model, has not yet been analyzed.

In earlier work [12] we analyzed the game dynamics of the SUM-SG and showed that if the initial network G_0 is a tree on n nodes, then the network creation process is guaranteed to converge in $\mathcal{O}(n^3)$ steps. It follows that the SUM-SG on trees is a poly-FIPG. It was shown that by employing the max cost policy, this process can be sped up significantly to $\mathcal{O}(n)$ steps, which is asymptotically optimal. For the SUM-SG on general graphs we showed that there exist initial networks for which there exists a cyclic sequence of best response moves. This implies that the SUM-SG on arbitrary initial networks is not a FIPG.

Very recently, Cord-Landwehr et al. [6] studied a variant of the MAX-SG, where agents have communication interests, and showed that this variant admits a best response cycle on a tree network as initial network. Hence the restricted-interest variant of the MAX-SG is not a FIPG – even on trees.

Brandes et al. [5] were the first to observe that the SUM-BG is not a FIPG and they prove this by providing a better response cycle. Very recently, Bilò et al. [4] gave a better response cycle for the MAX-BG which implies the same statement for this version. Note, that both proofs contain agents who perform a sub-optimal move at some step in the better response cycle. Hence, these two results do not address the convergence behavior if agents play optimally.

1.4 Our Contribution

In this work, we study Network Creation Games, as proposed by Fabrikant et al. [11], and several natural variants of this model from a new perspective. Instead of analyzing properties of equilibrium states, we apply a more constructive point of view by asking if and how fast such desirable states can be found by selfish agents. For this, we turn the original model and its variants, which are originally formulated as one-shot simultaneous-move games, into more algorithmic models, where moves are performed sequentially.

For the MAX Swap Game on trees, we show that the process must converge in $\mathcal{O}(n^3)$ steps, where n is the number of agents. Furthermore, by introducing a natural way of coordination we obtain a significant speed-up to $\Theta(n \log n)$ steps, which is almost optimal. We show that these results, combined with results from our earlier work [12], give the same bounds for the Asymmetric Swap Game on trees in both the SUM- and the MAX-version.

These positive results for initial networks which are trees are contrasted by several strong negative results on general networks. We show that the MAX-SG, the SUM-ASG and the MAX-ASG on general networks are *not* guaranteed to converge if agents repeatedly perform best possible improving moves and, even worse, that *no* move policy can enforce convergence. We show that these games are not in FIPG, which implies that there cannot exist a generalized ordinal potential function which “guides” the way towards an equilibrium state. For the SUM-

ASG we show the even stronger negative result that it can happen that *no* sequence of best response moves may enforce convergence, that is, the SUM-ASG is not even weakly acyclic under best response. If not all possible edges can be created, that is if we have a non-complete *host graph* [4, 7], then we show that the SUM-ASG and the MAX-ASG on non-tree networks is not weakly acyclic. Moreover, we map the boundary between convergence and non-convergence in ASGs and show the surprising result that cyclic behavior can already occur in n -vertex networks which have n edges. That is, even one non-tree edge suffices to completely change the dynamic behavior of these games. In our constructions we have that every agent owns exactly one edge, which is equivalent to the uniform-budget case introduced by Ehsani et al. [9]. In their paper [9] the authors raise the open problem of determining the convergence speed for the bounded-budget version. Thus, our results answer this open problem – even for the simplest version of these games – in the negative, since we show that no convergence guarantee exists.

Finally, we provide best response cycles for all versions of the Buy Game, which implies that these games have no convergence guarantee – even if agents have the computational resources to repeatedly compute best response strategies. To the best of our knowledge, the existence of *best* response cycles for all these versions was not known before. This addresses a proposed extension of the original model [11], where the networks evolve in stages with new players arriving in every stage and all stages should be equilibria. We show that even within such a stage the desired stabilization, apart from computational issues, may not happen.

	SUM-SG	SUM-ASG	BB SUM-(G)BG	SUM-(G)BG
Tree	poly-FIPG [12]	poly-FIPG (Cor.2)	–	BRC (Thm.11)
Non-Tree	BRC [12]	\notin BR-WAG (Thm.7)	BRC (Thm.9)	BRC (Thm.11)

Table 1: Summary of results for the SUM-version. BB abbreviates “bounded-budget” [9], BRC abbreviates “best response cycle”.

	MAX-SG	MAX-ASG	BB MAX-(G)BG	MAX-(G)BG
Tree	poly-FIPG (Thm.1)	poly-FIPG (Cor.2)	–	BRC (Thm.12)
Non-Tree	BRC (Thm.6), no move-policy	BRC (Thm.8), no move-policy	BRC (Thm.10)	BRC (Thm.12)

Table 2: Summary of results for the MAX-version.

2 Dynamics in Max Swap Games

In this section we focus on the game dynamics of the MAX-SG. Interestingly, we obtain results which are very similar to the results shown in our earlier work [12] but we need entirely different techniques to derive them. Omitted proofs can be found in the Appendix.

2.1 Dynamics on Trees

We will analyze the network creation process in the MAX-SG when the initial network is a tree. We prove that this process has the following desirable property:

Theorem 1. *The MAX-SG on trees is guaranteed to converge in $\mathcal{O}(n^3)$ steps to a stable network. That is, the MAX-SG on trees is a poly-FIPG.*

Before proving Theorem 1, we analyze the impact of a single edge-swap.

Let $T = (V, E)$ be a tree on n vertices and let agent v be unhappy in network T . Assume that agent v can decrease her cost by performing the edge-swap vu to vw , for some $u, w \in V$. This swap transforms T into the new network $T' = (V, (E \setminus \{vu\}) \cup \{vw\})$. Let $c_T(v) = \max_{x \in V(T)} d_T(v, x)$ denote agent v 's cost in the network T . Let $c_{T'}(u)$ denote her respective cost in T' . Let A denote the tree of $T'' = (V, E \setminus \{vu\})$ which contains v and let B be the tree of T'' which contains u and w . It is easy to see, that we have $d_T(x, y) = d_{T'}(x, y)$, if $x, y \in V(A)$ or if $x, y \in V(B)$.

Lemma 1. *For all $x \in V(A)$ there is no $y \in V(A)$ such that $c_T(x) = d_T(x, y)$.*

Proof. By assumption, agent v can decrease her cost by swapping the edge vu to edge vw , where $u, w \in V(B)$. We have that $d_T(x, v) < c_T(v)$, for all $x \in V(A)$, since otherwise this swap would not change agent v 's cost. It follows that for arbitrary $x, y \in V(A)$ we have

$$d_T(x, y) \leq d_T(x, v) + d_T(v, y) < d_T(x, v) + c_T(v).$$

Let $z \in V(B)$ be a vertex having maximum distance to v in T , that is, $c_T(v) = d_T(v, z)$. The above implies that $d_T(x, y) < d_T(x, z) = c_T(x)$, for all $x, y \in V(A)$. \square

The above lemma directly implies the following statement:

Corollary 1. *For all $x \in V(A)$, we have $c_T(x) > c_{T'}(x)$.*

Hence, we have that agent v 's improving move decreases the cost for all agents in $V(A)$. For agents in $V(B)$ this may not be true: The cost of an agent $y \in V(B)$ can increase by agent v 's move. Interestingly, the next result guarantees that such an increase cannot be arbitrarily high.

Lemma 2. *Let $x \in V(A)$ and $y \in V(B)$ such that $d_{T'}(x, y) = c_{T'}(y)$. It holds that $c_T(x) > c_{T'}(y)$.*

Proof. In tree T we have $c_T(x) = d_T(x, v) + d_T(u, z) + 1$. Furthermore, in tree T' we have $c_{T'}(y) = d_{T'}(x, v) + d_{T'}(w, y) + 1$. Since $c_T(v) > c_{T'}(v)$, it follows that $d_T(w, y) < d_T(u, z)$, where $z \in V(B)$ is a vertex having maximum distance to v in T . Hence, we have

$$c_T(x) - c_{T'}(y) = d_T(u, z) - d_T(w, y) > 0.$$

\square

Towards a generalized ordinal potential function we will need the following:

Definition 1 (Sorted Cost Vector and Center-Vertex). *Let G be any network on n vertices. The sorted cost vector of G is $\vec{c}_G = (\gamma_G^1, \dots, \gamma_G^n)$, where γ_G^i is the cost of the agent, who has the i -th highest cost in the network G . An agent having cost γ_G^n is called center-vertex of G .*

Lemma 3. *Let T be any tree on n vertices. The sorted cost vector of T induces a generalized ordinal potential function for the MAX-SG on T .*

Proof. Let v be any agent in T , who performs an edge-swap which strictly decreases her cost and let T' denote the network after agent v 's swap. We show that

$$c_T(v) - c_{T'}(v) > 0 \Rightarrow \vec{c}_T >_{\text{lex}} \vec{c}_{T'},$$

where $>_{\text{lex}}$ is the lexicographic order on \mathbb{N}^n . The existence of a generalized ordinal potential function then follows by mapping the lexicographic order on \mathbb{N}^n to an isomorphic order on \mathbb{R} .

Let the subtrees A and B be defined as above and let $c_T(v) - c_{T'}(v) > 0$. By Lemma 1 and Lemma 2, we know that there is an agent $x \in V(A)$ such that $c_T(x) > c_{T'}(y)$, for all $y \in V(B)$. By Lemma 1 and Corollary 1, we have that $c_T(x) > c_{T'}(x)$, which implies that $\vec{c}_T >_{\text{lex}} \vec{c}_{T'}$. \square

Lemma 4. *Let $T = (V, E)$ be a connected tree on n vertices having diameter $D \geq 4$. After at most $\frac{nD-D^2}{2}$ moves of the MAX-SG on T one agent must perform a move which decreases the diameter.*

For proving Lemma 4 we need two easy-to-see properties of the sorted cost vector and another observation about center-vertices.

Observation 2. *Let G be any connected network on n nodes and let $\vec{c}_G = (\gamma_G^1, \dots, \gamma_G^n)$ be its sorted cost vector. We have $\gamma_G^1 = \gamma_G^2$ and $\gamma_G^n = \left\lceil \frac{\gamma_G^1}{2} \right\rceil$.*

Definition 2 (Longest Path). *Let G be any connected network. Let v be any agent in G having cost $c_G(v) = k$. Any simple path in G , which starts at v and has length k is called a longest path of agent v .*

As we will see, center-vertices and longest paths are closely related.

Lemma 5. *Let T be any connected tree and let v^* be a center-vertex of T . Vertex v^* must lie on all longest paths of all agents in $V(T)$.*

Proof. Let P_{xy} denote the path from vertex x to vertex y in T . We assume towards a contradiction that there are two vertices $v, w \in V(T)$, where $c_T(v) = d_T(v, w)$, and that $v^* \notin V(P_{vw})$. Let $z \in V(T)$ be the only shared vertex of the three paths $P_{vv^*}, P_{wv^*}, P_{vw}$. We have $d_T(v, z) < d_T(v, v^*) \leq c_T(v^*)$ and $d_T(w, z) < d_T(w, v^*) \leq c_T(v^*)$. We show that $c_T(z) < c_T(v^*)$, which is a contradiction to v^* being a center-vertex in T .

Assume that there is a vertex $u \in V(T)$ with $d_T(u, z) \geq c_T(v^*)$. It follows that $V(P_{vz}) \cap V(P_{zu}) = \{z\}$, since otherwise $d_T(v^*, u) = d_T(v^*, z) + d_T(z, u) > c_T(v^*)$. But now, since $d_T(z, w) < c_T(v^*) \leq d_T(z, u)$, we have $d_T(v, u) > c_T(v)$, which clearly is a contradiction. Hence, we have $d_T(z, u) < c_T(v^*)$, for all $u \in V(T)$, which implies that $c_T(z) < c_T(v^*)$. \square

Proof of Lemma 4. Let $v, w \in V$ such that $d_T(v, w) = D \geq 4$ and let P_{vw} be the path from v to w in T . Clearly, if no agent in $V(P_{vw})$ makes an improving move, then the diameter of the network does not change. On the other hand, if the path P_{vw} is the unique path in T having length D , then any improving move of an agent in $V(P_{vw})$ must decrease the diameter by at least 1. The network creation process starts from a connected tree having diameter $D \geq 4$ and, by Lemma 3, must converge to a stable tree in a finite number of steps. Moreover, Lemma 3 guarantees that the diameter of the network cannot increase in any step of the process. It was shown by Alon et al. [2] that any stable tree has diameter at most 3. Thus, after a finite number of steps the diameter of the network must strictly decrease, that is, on all paths of length D some agent must have performed an improving move which reduced the length of the respective path. We fix the path P_{vw} to be the path of length D in the network which survives longest in this process.

It follows, that there are $|V \setminus V(P_{vw})| = n - (D + 1)$ agents which can perform improving moves without decreasing the diameter. We know from Observation 2 and Lemma 5 that each one of those $n - (D + 1)$ agents can decrease her cost to at most $\lceil \frac{D}{2} \rceil + 1$ and has to decrease her cost by at least 1 for each edge-swap. We show that an edge-swap of such an agent does not increase the cost of any other agent and use the minimum possible cost decrease per step to conclude the desired bound.

Let $u \in V(T) \setminus V(P_{vw})$ be an agent who decreases her cost by swapping the edge ux to uy and let T' be the tree after this edge-swap. Let $a, b \in V(T)$ be arbitrary agents. Clearly, if $\{u, y\} \not\subseteq V(P_{ab})$ in T' , then $d_T(a, b) = d_{T'}(a, b)$. Let A be the tree of $T'' = (V, E \setminus \{uy\})$ which contains u and let B be the tree of T'' which contains y . W.l.o.g. let $a \in V(A)$ and $b \in V(B)$. By Corollary 1, we have $c_T(z) > c_{T'}(z)$ for all $z \in V(A)$ and it follows that $V(A) \cap V(P_{vw}) = \emptyset$. Hence, it remains to analyze the change in cost of all agents in $V(B)$.

If no vertex on the path P_{ab} is a center-vertex in T' , then, by Lemma 5, we have that $d_{T'}(a, b) < c_{T'}(b)$. It follows that every longest path of agent b in T' lies entirely in subtree B which implies that $c_{T'}(b) \leq c_T(b)$.

If there is a center-vertex of T' on the path P_{ab} in T' , then let v^* be the last such vertex on this path. We have assumed that the diameters of T' and T are equal, which implies that P_{vw} is a longest path of agent v in T' . Since, by Lemma 5, any center-vertex of T' must lie on all longest paths, it follows that v^* is on the path P_{vw} and we have $v^* \in V(B)$. W.l.o.g. let $d_{T'}(v, b) \geq d_{T'}(w, b)$. We have $d_{T'}(a, b) = d_{T'}(a, v^*) + d_{T'}(v^*, b) \leq d_{T'}(v, v^*) + d_{T'}(v^*, b)$. Hence, we have $d_{T'}(a, b) \leq c_{T'}(b)$. Since the path P_{bv} is in subtree B , we have $c_{T'}(b) \leq c_T(b)$.

Now we can easily conclude the upper bound on the number of moves which do not decrease the diameter of T . Each of the $n - (D + 1)$ agents with cost at most D may decrease their cost to $\lceil \frac{D}{2} \rceil + 1$. If we assume a decrease of 1 per step, then this yields the following bound:

$$(n - (D + 1)) \left(D - \left(\left\lceil \frac{D}{2} \right\rceil + 1 \right) \right) < (n - D) \frac{D}{2} = \frac{nD - D^2}{2}.$$

□

Finally, we can set out to proving Theorem 1.

Proof of Theorem 1. By Lemma 3, we know there exists a generalized ordinal potential function for the MAX-SG on trees. Hence, we know that this game is a FIPG and we are left

to bound the maximum number of improving moves needed for convergence. It was already shown by Alon et al. [2], that the only stable trees of the MAX-SG on trees are stars or double-stars. Hence, the process must stop at the latest when diameter 2 is reached.

Let $N_n(T)$ denote the maximum number of moves needed for convergence in the MAX-SG on the n -vertex tree T . Let $D(T)$ be the diameter of T . Let $D_{i,n}$ denote the maximum number of steps needed to decrease the diameter of any n -vertex tree having diameter i by at least 1. Hence, we have

$$N_n(T) \leq \sum_{i=3}^{D(T)} D_{i,n} \leq \sum_{i=3}^{n-1} D_{i,n},$$

since the maximum diameter of a n -vertex tree is $n - 1$. By applying Lemma 4 and adding the steps which actually decrease the diameter, this yields

$$N_n(T) \leq \sum_{i=3}^{n-1} D_{i,n} < \sum_{i=3}^{n-1} \left(\frac{ni - i^2}{2} + 1 \right) < n + \frac{n}{2} \left(\sum_{i=1}^n i \right) - \frac{1}{2} \left(\sum_{i=1}^n i^2 \right) \in \mathcal{O}(n^3).$$

□

Now let us analyze the impact of enforced coordination by a suitable move policy. We have the following result, which is close to optimal, since it is easy to see that there are instances in which $\Omega(n)$ steps are necessary:

Theorem 3. *The MAX-SG on trees with the max cost policy as move policy converges in $\Theta(n \log n)$ moves.*

We prove Theorem 3, by proving the lower and the upper bound separately, starting with the former. Since we analyze the max cost policy, we need two additional observations.

Observation 4. *An agent having maximum cost in a tree T must be a leaf of T .*

Observation 5. *Let u be an unhappy agent in $T = (V, E)$ and let u be a leaf of T and let v be u 's unique neighbor. Let B be the tree of $T' = (V, E \setminus \{uv\})$ which contains v . The edge-swap uv to uw , for some $w \in V(B)$ is a best possible move for agent u if w is a center-vertex of B .*

Lemma 6. *There is a tree T on n vertices where the MAX-SG on T with the max cost policy needs $\Omega(n \log n)$ moves for convergence.*

Proof. We consider the path on n -vertices $P_n = v_1 v_2 \dots v_n$ of length $n - 1$. We apply the max cost policy and for breaking ties we will always choose the vertex having the smallest index among all vertices having maximum cost. If a maximum cost vertex has more than one best response move, then we choose the edge-swap towards the new neighbor having the smaller index. With these assumptions and with Observation 4 and Observation 5, we have that the center-vertex having the smallest index will “shift” towards a higher index, from $v_{\lceil n/2 \rceil}$ to v_{n-2} . Finally, agent v_n is the unique agent having maximum cost and her move transforms the tree to a star. See Fig. 1 for an illustration for $n = 9$.

We start by analyzing the change in costs of agent v_1 . Clearly, $c_0 = c_{P_n}(v_1) = n - 1$. By Observation 5, we know that v_1 's best swap connects to the minimum index center-vertex of the tree without vertex v_1 . Hence after the best move of v_1 this agent has cost

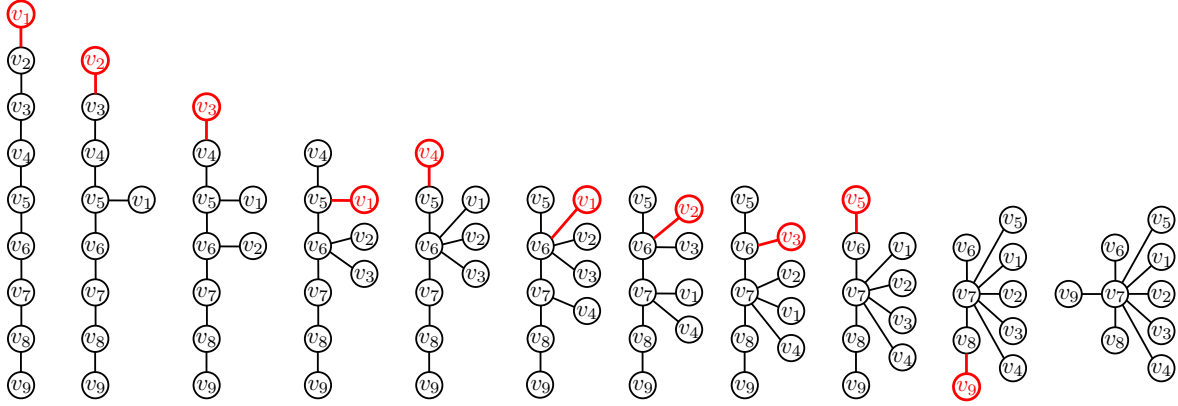


Figure 1: The convergence process for with $n = 9$ in the MAX-SG on P_n .

$c_1 = \lceil \frac{c_0-1}{2} \rceil + 1 > \frac{c_0}{2}$. When v_1 is chosen to move again, her cost can possibly decrease to $\lceil \frac{c_1-1}{2} \rceil + 1 > \frac{c_0}{4}$. After the i -th move of v_1 her cost is at least $\lceil \frac{c_i-1-1}{2} \rceil + 1 > \frac{c_0}{2^i}$. Thus, the max cost policy allows agent v_1 to move at least $\log \frac{c_0}{3}$ times until she is connected to vertex v_{n-2} , the center of the final star, where she has cost 3.

The above implies, that the number of moves of every agent allowed by the max cost policy only depends on the cost of that agent when she first becomes a maximum cost agent. Moreover, since all moving agents are leaves, no move of an agent increases the cost of any other agent. By construction, the cost of every moving agent is determined by her distance towards vertex v_n . Since agent v_n does not move until in the last step of the process, we have that a move of agent v_i does not change the cost of any other agent $v_j \neq v_n$ who moves after v_i . It follows, that we can simply add up the respective lower bounds on the number of moves of all players, depending on the cost when they first become maximum cost agents. It is easy to see, that agent v_i becomes a maximum cost agent, when the maximum cost is $n - i$. Let $M(P_n)$ denote the number of moves of the MAX-SG on P_n with the max cost policy and the above tie-breaking rules. This yields

$$M(P_n) > \sum_{c_0=n-1}^4 \log \frac{c_0}{3} \in \Omega(n \log n).$$

□

Lemma 7. *The MAX-SG on a n -vertex tree T with the max cost policy needs $\mathcal{O}(n \log n)$ moves to converge to a stable tree.*

Proof. Consider any tree T on n vertices. By Observation 4, we know that only leaf-agents are allowed to move by the max cost policy, which implies that no move of any agent increases the cost of any other agent. Observation 5 guarantees that the best possible move of a leaf-agent u having maximum cost c decreases agent u 's cost to at most $\lceil \frac{c}{2} \rceil + 1$. Hence, after $\mathcal{O}(\log n)$ moves of agent u her cost must be at most 3. If the tree converges to a star, then agent u may move one more time. If we sum up over all n agents, then we have that after $\mathcal{O}(n \log n)$ moves the tree must be stable. □

2.2 Dynamics on General Networks

In this section we show that allowing cycles in the initial network completely changes the dynamic behavior of the MAX-SG.

Theorem 6. *The MAX-SG on general networks admits best response cycles. Moreover, no move policy can enforce convergence. The first result holds true even if agents are allowed to swap more than one edge at a time.*

Proof of Theorem 6. We prove the theorem by showing that there exists an initial network which induces a best response cycle and where in every step of the cycle exactly one agent is unhappy. The existence of the best response cycle shows that the MAX-SG on this instance does not have the finite improvement property. The fact that in every step of the cycle exactly one agent is unhappy shows that no move policy can avoid that cyclic behavior. In every step, swapping one edge suffices to achieve the best possible cost decrease for the moving agent. Hence, there exists a best response cycle even if agents are allowed to perform multi-swaps. However, note that with multi-swaps it is no longer true that there is only one unhappy agent in every step.

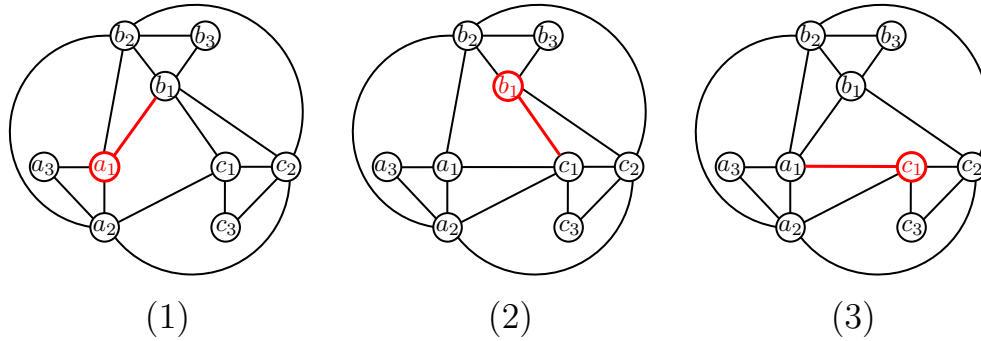


Figure 2: The steps of a best response cycle for the MAX-SG on general networks.

Consider the initial network G_1 which is depicted in Fig. 2 (1). Note, that only agents a_1, a_3, b_3 and c_3 have cost 3 while all other agents have cost 2. Clearly, agents having cost 2 cannot improve on their current situation. Agents a_3, b_3, c_3 cannot perform an improving move, since all of them have exactly two vertices in distance 3 and there is no vertex which is a neighbor of both of them. This leaves agent a_1 as the only possible candidate for an improving edge-swap. A best possible move for agent a_1 is the swap a_1b_1 to a_1c_1 , which yields a distance decrease of 1 which is clearly optimal. This swap transforms G_1 into G_2 , which is depicted in Fig. 2 (2). Observe, that G_2 is isomorphic to G_1 , with agent b_1 facing exactly the same situation as agent a_1 in G_1 . Agent b_1 has the swap b_1c_1 to b_1a_1 as best response move and we end up in network G_3 , shown in Fig. 2 (3). Again, G_3 is isomorphic to G_1 , now with agent c_1 being the unique unhappy agent. Agent c_1 's best possible swap transforms G_3 back into G_1 . \square

3 Dynamics in Asymmetric Swap Games

In this section we consider the SUM-ASG and the MAX-ASG. Note, that now we assume that each edge has an owner and only this owner is allowed to swap the edge. We show that we can directly transfer the results from above and from [12] to the asymmetric version if the initial network is a tree. On general networks we show even stronger negative results.

Observe, that the instance used in the proof of Theorem 6 and the corresponding instance in [12] show that best response cycles in the Swap Game are not necessarily best response cycles in the Asymmetric Swap Game. We will show the rather counter-intuitive result that this holds true for the other direction as well. Hence, in general we cannot simply transfer the cyclic instances from one model to the other even though they are closely related.

3.1 Asymmetric Swap Games on Trees

The results in this section follow from the respective theorems in [12] and from the results in Section 2.1 and are therefore stated as corollaries.

Corollary 2. *The SUM-ASG and the MAX-ASG on n -vertex trees are both a poly-FIPG and both must converge to a stable tree in $\mathcal{O}(n^3)$ steps.*

Proof. It was shown in [12] that the SUM-SG on trees is an ordinal potential game, where the social cost, which is the sum of all agent's costs, serves as ordinal potential function. Furthermore, it was shown that the SUM-SG on n -vertex trees must converge in $\mathcal{O}(n^3)$ steps. Note, that ordinal potential games are a subclass of FIPG [16].

The only difference in the *asymmetric* version of this game is that edges have owners and only the respective owner is allowed to swap an edge. Clearly, since any improving swap decreases the value of the (generalized) potential function, this is independent of the edge-ownership. Furthermore, with edges having owners, we have that in each network less moves are possible and every moving agent has, compared with the Swap Game, at most the same number of admissible strategies in any step. Thus, the convergence process cannot be slower. The results from [12] and Theorem 1 then yields the desired statement. \square

Corollary 3. *Using the max cost policy and assuming a n -vertex tree as initial network, we have that*

- *the SUM-ASG converges in $\max\{0, n-3\}$ steps, if n is even and in $\max\{0, n + \lceil n/2 \rceil - 5\}$ steps, if n is odd. Moreover, both bounds are tight and asymptotically optimal.*
- *the MAX-ASG converges in $\Theta(n \log n)$ steps.*

Proof. We can carry over the results from [12] and Section 2.1 about speeding up the convergence process by a suitable move policy. The reason for this is that in all used lower bound constructions it holds that whenever an edge is swapped more than once, then it is the same incident agent who moves again. Hence, we can assign the edge-ownership to this agent and get the same lower bounds in the asymmetric version. The upper bounds carry over trivially, since agents cannot have more admissible new strategies in any step in the asymmetric version compared to the version without edge-owners. \square

3.2 Sum Asymmetric Swap Games on General Graphs

If we move from trees to general initial networks, we get a very strong negative result for the SUM-ASG: There is no hope to enforce convergence if agents stick to playing best responses even if multi-swaps are allowed.

Theorem 7. *The SUM-ASG on general networks is not weakly acyclic under best response. Moreover, this result holds true even if agents can swap multiple edges in one step.*

Proof. We give a network which induces a best response cycle. Additionally, we show that in each step of this cycle exactly one agent can decrease her cost by swapping an edge and that the best possible swap for this agent is unique in every step. Furthermore, we show that the moving agent cannot outperform the best possible single-swap by a multi-swap. This implies that if agents stick to best response moves then *no* best response dynamic can enforce convergence to a stable network and allowing multi-swaps does not alter this result.

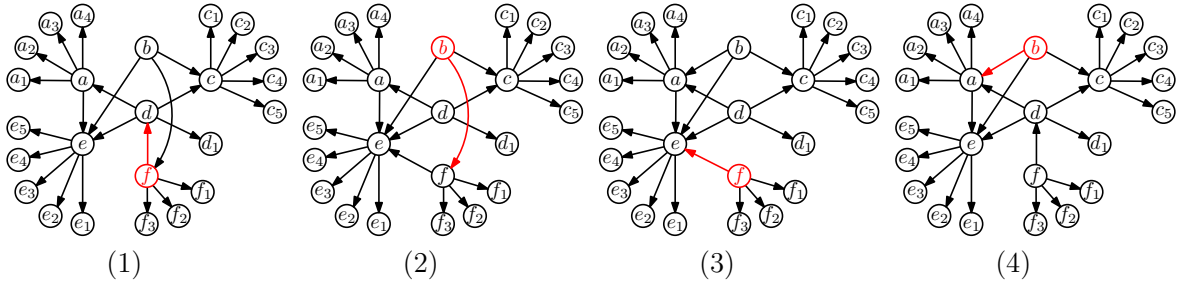


Figure 3: The steps of a best response cycle for the SUM-ASG on general networks. Note, that edge directions indicate edge-ownership. All edges are two-way.

The best response cycle consists of the networks G_1, G_2, G_3 and G_4 given in Fig. 3. We begin with showing that in G_1, \dots, G_4 all agents, except agent b and agent f , cannot perform an improving strategy change even if they are allowed to swap multiple edges in one step.

In G_1, \dots, G_4 all leaf-agents do not own any edges and the agents c and e cannot swap an edge since otherwise the network becomes disconnected. For the same reason, agent d cannot move the edge towards d_1 . Agent d owns three other edges, but they are optimally placed since they are connected to the vertices having the most leaf-neighbors. It follows, that agent d cannot decrease her cost by swapping one edge or by performing a multi-swap. Note, that this holds true for all networks G_1, \dots, G_4 , although the networks change slightly. Agent a cannot move her edges towards a_i , for $1 \leq i \leq 4$. On the other hand, it is easy to see that agent a 's edge towards vertex e cannot be swapped to obtain a strict cost decrease since the most promising choice, which is vertex c , yields the same cost in G_1 and G_4 and even higher cost in G_2 and G_3 . Trivially, no multi-swap is possible for agent a .

Now, we consider agent b and agent f . First of all, observe that in G_1, \dots, G_4 agent f owns exactly one edge which is not a bridge. Thus, agent f cannot perform a multi-swap in any step of the best response cycle. Agent b , although owning three edges, is in a similar situation: Her edges to vertex c and e can be considered as fixed, since swapping one or both of them does not yield a cost decrease in G_1, \dots, G_4 . Hence, agent b and agent f each have one “free” edge to operate with. In G_1 agent b 's edge towards f is placed optimally,

since swapping towards a or d does not yield a cost decrease. In G_3 , agent b 's edge towards a is optimal, since swapping towards d or f does not decrease agent b 's cost. Analogously, agent f 's edge towards e in G_2 and her edge towards d in G_4 are optimally placed.

Last, but not least, we describe the best response cycle: In G_1 agent f can improve and her unique best possible edge-swap in G_1 is the swap from d to e , yielding a cost decrease of 4. In G_2 agent b has the swap from f to a as unique best improvement which yields a cost decrease of 1. In G_3 agent f being unhappy with her strategy and the unique best swap is the one from e to d yielding an improvement of 1. In G_4 it is agent b 's turn again and her unique best swap is from a to f which decreases her cost by 3. After agent b 's swap in G_4 we arrive again at network G_1 , hence G_1, \dots, G_4 is a best response cycle where in each step exactly one agent has a single-swap as unique best possible improvement. \square

Remark 1. *Note, that the best response cycle presented in the proof of Theorem 7 is not a best response cycle in the SUM-SG. The swap fb to fe of agent f in G_1 yields a strictly larger cost decrease than her swap fd to fe .*

Corollary 4. *The SUM-ASG on a non-complete host graph is not weakly acyclic.*

Proof. We use the best response cycle G_1, \dots, G_4 shown in Fig. 3 and let the host graph H be the complete graph but without the edge $\{a, f\}$. In this case, agent f 's best response move in G_1 is her only possible improving move. For the networks G_2, G_3 and G_4 it is easy to check, that the respective moving player has exactly one possible improving move. \square

3.3 Max Asymmetric Swap Games on General Graphs

Compared to Theorem 7, we show a slightly weaker negative result for the max-version of Asymmetric Swap Games.

Theorem 8. *The MAX-ASG on general networks admits best response cycles. Moreover, no move policy can enforce convergence.*

Proof. We show that there exists a best response cycle for the MAX-ASG, where no move policy can enforce convergence. Our cycle, shown in Fig. 4, has six steps G_1, \dots, G_6 . In G_3 and G_6 there are two unhappy agents whereas in the other steps there is exactly one unhappy agent. It turns out, that independently which one of the two agent moves in G_3 or G_6 , there is a best response move which leads back to a network in the cycle. This implies, that no move policy may enforce convergence.

First of all, note that the networks G_2 and G_5 are isomorphic. The same holds true for the networks G_3 and G_6 . We start by showing that in the networks G_1, \dots, G_4 only the highlighted agents are unhappy. Then we will analyze the best response moves of the unhappy agents in the respective networks.

Consider any G_i , where $1 \leq i \leq 4$. Clearly, no leaf-agent of G_i can perform any swap. Agent g has cost 3 in G_i . We have $d_{G_i}(g, l_1) = d_{G_i}(g, l_5)$ and there is no vertex in G_i which is a neighbor to both l_1 and l_5 . Thus, agent g cannot achieve cost 2, which implies that agent g does not want to perform a move in G_i . By the same argument, it follows that any agent having cost 3 must be happy. Hence, we have that agent b in G_1 and G_2 and agent c in G_3 and G_4 do not swap.

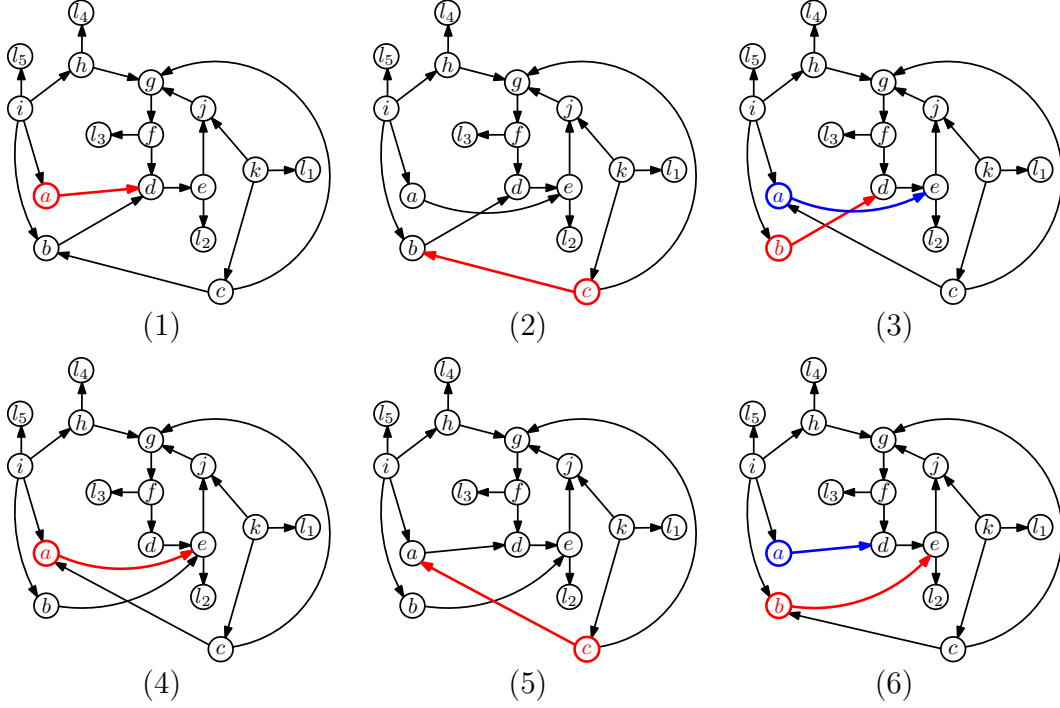


Figure 4: The steps of a best response cycle for the MAX-ASG on general networks.

Agent d , having cost 4 in G_i , cannot perform an improving move in G_i , since $d_{G_i}(d, l_1) = d_{G_i}(d, l_4) = 4$ and any such improving move must connect to a vertex which has distance at most 2 to vertex l_1 . These vertices are c, j, k and l_1 but any of them has distance at least 3 towards l_4 in G_i . Agents f and i , both having cost 4 in G_i , each face an analogous situation, since $d_{G_i}(f, l_1) = d_{G_i}(i, l_1) = d_{G_i}(f, l_5) = d_{G_i}(i, l_3) = 4$ and since vertices c, j, k, l_1 have distance at least 3 to l_5 or l_3 in any G_i .

Agent h , having cost 4 in G_i has both l_1 and l_2 in distance 4. The only vertex which has distance at most 2 to both of them is vertex j . But if h swaps towards vertex j , then this yields distance 4 towards vertex l_3 .

Now we consider agent e . Any strategy yielding cost at most 3 for agent e must connect to c, j, k or l_1 , since otherwise vertex l_1 would be in distance 4. But, since $d_{G_i}(e, l_4) = 4$ and since c, j, k, l_1 all have distance at least 3 towards l_4 , no such strategy can exist. For agent j , having cost 4 in G_i , the situation is similar. Any strategy having cost at most 3 for agent j must connect to a, b, h, i or l_5 , since otherwise j 's distance towards l_5 would be 4. But if j swaps away from g to any vertex in $\{a, b, h, i, l_5\}$, then her distance to l_3 increases to 4.

Agent k , having cost 4 in G_i , has vertex l_4 and l_5 in distance 4. Thus, to achieve as cost of at most 3, agent k must connect to vertex h or i . Both h and i have distance at least 3 towards l_3 . Since $d_{G_i}(k, l_3) = 4$, it follows, that agent k cannot perform an improving move.

Now, we are left with agent c in G_1 , agent a in G_2 and agent b in G_4 , all having cost 4. We have $d_{G_1}(c, l_2) = 4$. Thus, any strategy which yields cost at most 3 must connect to d, e, j or l_2 . If c swaps away from g , then her distance to l_4 increases to 4. If c swaps away from b , then her distance to l_5 increases to 4. Hence, agent c cannot swap an edge to

decrease her cost in G_1 . For agent a in G_2 and agent b in G_4 the situation is similar. We have $d_{G_2}(a, l_1) = d_{G_2}(a, l_3) = d_{G_4}(b, l_1) = d_{G_4}(b, l_3) = 4$. Both agents cannot decrease their cost by swapping an edge in the respective network, since all vertices which have distance at most 2 to l_1 have distance at least 3 to l_3 .

Finally, we analyze the best response moves of all unhappy agents in any G_i . In G_1 only agent a , having cost 5, is unhappy. There is only one vertex having distance 5 to a , which is l_1 . Thus, agent could swap towards any vertex having distance at most 3 to l_1 to improve on this distance. Possible target-vertices are b, c, e, g, j, k and l_1 . But, after any such swap, agent a must have distance 4 to vertex l_3 , which implies, that agent a cannot decrease her cost by more than 1. Furthermore, swapping towards c, k or l_1 yields distance 5 towards l_3 , which rules out these target vertices. If agent a performs the swap ad to ae in G_1 , then we obtain network G_2 . In G_2 only vertex c , having cost 4, is unhappy. Her unique vertex in distance 4 is l_2 . Thus, a swap towards a, d, e, j or l_2 would reduce this distance. However, only the swap towards a is an improving move, since in all other cases agent c 's distance to l_6 increases to 4. This swap transforms G_2 into G_3 . In G_3 we have that agent a and agent b are unhappy. Agent b in G_3 is in a similar situation as agent a in G_1 . Her only vertex in distance 5 is the leaf l_1 and by swapping towards a, c, e, g, j, k or l_1 this distance can be reduced. But any such swap yields that agent b 's distance to l_3 increases to at least 4, which implies that a cost decrease by 1 is optimal. Furthermore, the swaps towards c, k or l_1 are not improving moves since these yield distance 5 towards l_3 . If agent b performs the swap bd to be we obtain network G_4 . Agent a in G_3 has l_3 as her only vertex in distance 4. There is exactly one swap for agent a , which decreases this distance to 3 without increasing any other distance to more than 3, and this is the swap ae to ad . Note, that this swap of agent a transforms G_3 to a network which is isomorphic to network G_1 . Finally, we argue for agent a in G_4 . This agent is in a similar situation as agent a in G_3 . Her only vertex in distance 4 is l_3 and the swap towards d is the only swap which does not increase any other distance to more than 3. Thus, this move is agent a 's unique best response in G_4 and this move leads to network G_5 . \square

Corollary 5. *The MAX-ASG on a non-complete host graph is not weakly acyclic.*

Proof. We use the best response cycle G_1, \dots, G_6 from Fig. 4. As host graph H we use the complete graph, but without edges $\{a, b\}, \{a, g\}, \{a, j\}, \{b, g\}, \{b, j\}$. By inspecting the proof of Theorem 8, it is easy to see that in each step of the cycle the moving agent has exactly one improving move. \square

3.4 The Boundary between Convergence and Non-Convergence

In this section we explore the boundary between guaranteed convergence and cyclic behavior. Quite surprisingly, we can draw a sharp boundary by showing that the undesired cyclic behavior can already occur in n -vertex networks having exactly n edges. Thus, one non-tree edge suffices to radically change the dynamic behavior of Asymmetric Swap Games. Our constructions are such that each agent owns exactly one edge, which corresponds to the uniform unit budget case, recently introduced by Ehsani et al. [9]. Hence, even if the networks are build by identical agents having a budget the cyclic behavior may arise. This answers the open problem in [9] the negative.

Theorem 9. *The SUM-ASG admits a best response cycle on a network where every agent owns exactly one edge.*

Proof. The network which induces a best response cycle and the steps of the cycle are shown in Fig. 5. Let n_k denote the number of vertices having the form k_j , for some index j .

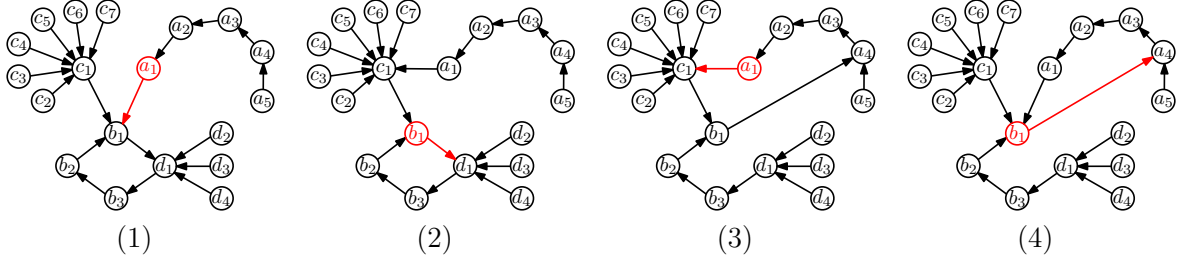


Figure 5: The steps of a best response cycle for the SUM-ASG where each agent owns exactly one edge.

In the first step, depicted in Fig. 5 (1), agent a_1 has only one improving move, which is the swap from b_1 to c_1 . This swap reduces agent a_1 's cost by 1, since $n_c = n_b + n_d + 1$. After this move, shown in Fig. 5 (2), agent b_1 is no longer happy with her edge towards d_1 , since by swapping towards a_4 she can decrease her cost by 2. This is a best possible move for agent b_1 (note, that a swap towards a_3 yields the same cost decrease). But now, in the network shown in Fig. 5 (3), by swapping back towards vertex b_1 , agent a_1 can additionally decrease her distances to vertices a_4 and a_5 by 1. This yields that agent a_1 's swap from c_1 to b_1 decreases her cost by 1. This is true, since all distances to c_j vertices increase by 1 but all distances to b_i and d_i vertices and to a_4 and a_5 decrease by 1 and since we have $n_c = n_b + n_d + 1$. Note, that this swap is agent a_1 's unique improving move. By construction, we have that after agent a_1 has swapped back towards b_1 , depicted in Fig. 5 (4), agent b_1 's edge towards a_4 only yields a distance decrease of 7. Hence, by swapping back towards d_1 , agent b_1 decreases her cost by 1, since her sum of distances to the d_j vertices decreases by 8. This swap is the unique improving move of agent b_1 in this stage. Now the best response cycle starts over again, with agent a_1 moving from b_1 to c_1 . \square

Theorem 10. *The MAX-ASG admits a best response cycle on a network where every agent owns exactly one edge.*

Proof. Fig. 6 shows the steps of a best response cycle for the MAX version in a network, where each agent owns exactly one edge.

In the first step of the cycle, shown in Fig. 6 (1), agent a_1 can decrease her maximum distance from 6 to 5 by swapping from e_1 to one of the vertices e_2, \dots, e_5 . Note, that all these swaps yield the same distance decrease of 1 and, since a_1 has distance 5 towards a_6 and swapping towards any of the a_i -vertices is obviously sub-optimal, no other swap can yield a larger cost decrease. By agent a_1 performing the swap towards e_5 we obtain the network in Fig. 6 (2).

Now, agent b_1 can improve her situation with a swap from a_1 to a_2 or to a_3 . Both possible swaps reduce her maximum distance from 6 to 5. This is best possible: The cycle has length

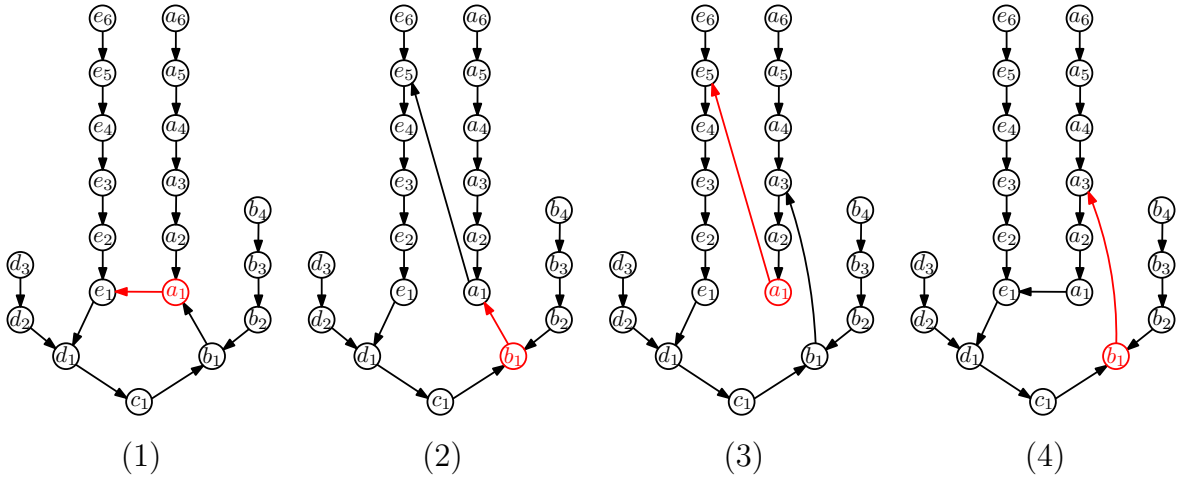


Figure 6: The steps of a best response cycle for the MAX-ASG where each agent owns exactly one edge.

9 before agent b_1 's move, which implies that there are two vertices on the cycle which have distance 4 to b_1 . Observe, that agent b_1 must swap towards a vertex which has at most distance 4 to vertex a_6 to reduce her maximum distance. Clearly, only one of the a_i vertices, with $i \neq 1$, is possible. However, swapping away from a_1 must increase the cycle by at least 1, which implies that after this swap agent b_1 must have at least one cycle-vertex in distance 5. Hence, no swap can decrease agent b_1 's maximum distance by more than 1. Let agent b_1 perform the swap towards a_3 and we end up with the network in Fig. 6 (3).

Agent a_1 now finds herself sitting in a large cycle having maximum distance 7 towards d_3 and distance 6 to vertex b_4 . By swapping from e_5 to one of the vertices e_1, e_2, e_3 , agent a_1 can reduce her maximum distance to 6. This is optimal, since all improving moves must swap towards a vertex having at most distance 5 to vertex d_3 and agent a_1 cannot move to far away from vertex e_6 . The vertices e_1, e_2, e_3 are the only vertices which satisfy both conditions. Let a_1 swap towards e_1 and we get the network depicted in Fig. 6 (4).

In the last step of the best response cycle, we have agent b_1 with maximum distance 8 towards vertex e_6 . Clearly, agent b_1 wants to move closer to this vertex but this implies, that she must move away from vertex a_6 . The only possible compromise between both distances is a swap either to a_1 or to e_1 . Both these swaps yield a decrease of b_1 's maximum distance by 1. By swapping towards vertex a_1 , we end up with our starting configuration and the cycle is complete. \square

Remark 2. We can give best response cycles for both versions of the ASG for the case where every agent owns exactly two edges. We conjecture, that such cyclic instances also exist for all cases where every agent owns exactly k edges, for any $k \geq 3$. In particular, it would be interesting if there is a generic construction which works for all $k \geq 1$.

4 Dynamics in (Greedy) Buy Games

We focus on the dynamic behavior of the Buy Game and the Greedy Buy Game. Remember, that we assume, that each edge can be created for the cost of $\alpha > 0$. We show that best response cycles exist, even if arbitrary strategy-changes are allowed. However, on the positive side, we were not able to construct best response cycles where only one agent is unhappy in any step. Hence, the right move policy may have a substantial impact in (Greedy) Buy Games.

All the above constructions of best response cycles cannot be used for our purpose, since α has to be chosen high enough such that no agent wants to buy any additional edge, but at the same time low enough, such that all cycles in the network remain in place.

Theorem 11. *The SUM-GBG and the SUM-BG admit best response cycles.*

Proof. We prove both statements by giving a best response cycle, where the best response of any moving agent consists of either buying, deleting or swapping one edge. The best response cycle G_1, \dots, G_6 , for $7 < \alpha < 8$, is depicted in Fig. 7.

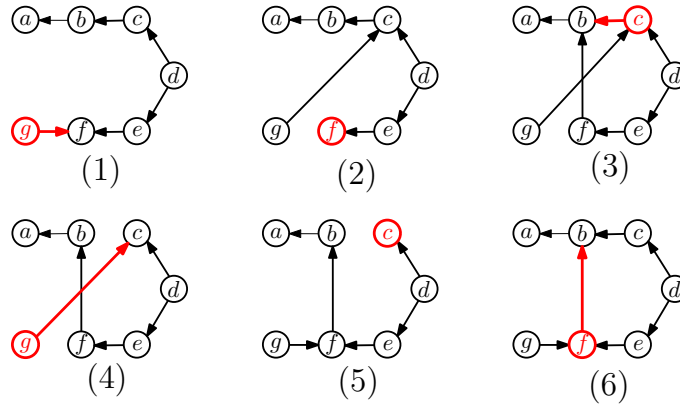


Figure 7: The steps of a best response cycle for the SUM-(G)BG for $7 < \alpha < 8$.

We analyze the steps of the cycle and show, that the indicated strategy-change is indeed a best response move – even if there are no restrictions on the admissible strategies.

In network G_1 , it is obvious, that agent g is unhappy with her situation. The indicated swap gf to gc decreases agent g 's cost from 21 to 15. This is a best possible move, which can be seen as follows. Clearly, deleting her unique own edge would disconnect the network. Hence in all optimal strategies agent g must purchase at least one edge. Among all strategies, where g buys exactly one edge, that is, among all possible single edge-swaps, we have that buying an edge towards a vertex having minimum cost in $G_1 - g$ is optimal. Here, $G_1 - g$ is the network G_1 with vertex g removed. Thus, swapping her edge towards vertex c is optimal. This swap yields a cost of 15 for agent g . It is easy to see, that no other arbitrary strategy can be better, since if agent g buys at least two edges, then her cost is greater than 16, since $\alpha > 7$. After g has performed her strategy-change we obtain network G_2 .

In G_2 we claim that agent f is unhappy and that her best possible move is to buy an edge towards vertex b . First of all, this is an improving move, since the edge fb decreases

her cost from 19 to $11 + \alpha$, which is a strict cost decrease since $\alpha < 8$. The target vertex b is optimal, since connecting to c yields the same cost and connecting to any other vertex yields a higher cost. Clearly, agent f cannot delete or swap any edges. Furthermore, buying at least two edges yields cost of more than 19, since $2\alpha > 14$ and there are six other vertices in G_2 to which f has distance at least 1. The edge purchase of agent f leads us to network G_3 .

In network G_3 we claim that agent c is unhappy and that her best possible move is to delete her edge towards b . Agent c has cost $9 + \alpha$ in G_3 . Deleting edge cb yields cost $16 < 9 + \alpha$, since $\alpha > 7$. Clearly, no strategy which buys at least two edges can be optimal for agent c , since $6 + 2\alpha > 16$. On the other hand, swapping her unique edge away from b must increase agent c 's cost since at least one distance increases to 3. If agent c deletes her edge cb , then we obtain network G_4 .

In G_4 , we have that agent g is in a similar situation as she was in G_1 . Agent g is again a leaf-vertex of a path of length 6. Thus, by an analogous argument as for g in G_1 , we have that the swap gc to gf is a best possible move for agent g in G_4 . This move leads us to network G_5 .

In network G_5 we have that agent c is in a similar situation as agent f in G_2 . By analogous arguments, it follows that buying the edge towards b is a best possible move of agent c in G_5 . This edge-purchase transforms G_5 into G_6 .

Finally, in network G_6 we have that agent f is in a similar situation as agent c in G_3 . Thus, by analogous arguments, we have that deleting her edge fb is an optimal move for agent f in G_6 . This deleting transforms G_6 into G_1 and we have completed the cycle. \square

Theorem 12. *The MAX-GBG and the MAX-BG admit best response cycles.*

Proof. We give a best response cycle, where in every step of the cycle the moving agent has a best response which consists of a greedy move, that is, the strategy-change is the addition, deletion or swap of one edge. The best response cycle G_1, \dots, G_4 , for $1 < \alpha < 2$, can be seen in Fig. 8.

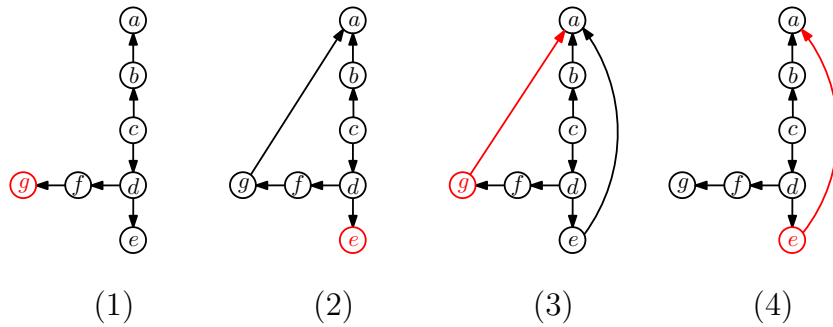


Figure 8: The steps of a best response cycle for the MAX-(G)BG for $1 < \alpha < 2$.

We show for each step of the cycle, that the indicated moving agent indeed performs a best response move which transforms the network of one step into the network of the next step of the cycle.

In network G_1 we claim that agent g is unhappy and that a best response move is to buy the edge ga . Clearly, agent g cannot delete or swap any edges. Hence, it suffices to analyze all buy or multi-buy operations. Agent g has cost 5 in G_1 . The purchase of edge ga is an improving move, since with this yields a distance cost of 3 for agent g and since $\alpha < 2$. Note, that agent g must buy at least one edge to reduce her distance cost in G_1 . Furthermore, it is easy to see that with one additional edge a distance cost of 3 is best possible. Now, observe, that if agent g buys more than one edge, then her distance cost may decrease, but it cannot decrease by more than 1 per edge. Since $\alpha > 1$, no strategy which buys at least two edges can yield strictly less cost than $3 + \alpha$. The indicated move of agent g transforms G_1 into G_2 .

In G_2 , agent e , having cost 4, is unhappy with her situation. Buy buying the edge ea , agent e can decrease her distance cost to 2. Since $\alpha < 2$, it follows that this is a improving move. Note, that agent e cannot delete or swap any edges and that distance cost of 2 is optimal, unless agent e buys 5 edges, which clearly is too expensive. Hence, buying the edge ea is a best response move for agent e and this move leads to network G_3 .

In network G_3 , we have that agent g , having cost $3 + \alpha$, is unhappy. By deleting her own edge ga , agent g can achieve a cost of 4, which is strictly less than $3 + \alpha$, since $\alpha > 1$. We claim that deleting edge ga is a best response move for agent g . If agent g swaps her unique own edge, then she cannot achieve a distance cost of less than 3. Thus, no swap can be an improving move. If agent g buys at least one additional edge, then such a purchase may decrease agent g 's distance cost by 1 per edge, but since $\alpha > 1$, this cannot outperform her current strategy in G_3 . Thus, deleting edge ga is the only improving move of agent g and, thus, must be her best response move. This move transforms network G_3 in to G_4 .

Finally, in network G_4 we have that agent e , having cost $3 + \alpha$, is unhappy. Deleting her edge ea yields a cost of 4, which is strictly less than $3 + \alpha$. Swapping this edge cannot decrease her distance cost below 3, which rules out any edge-swaps. If agent e buys at least one additional edge, then she can reduce her distance cost by at most 1 per additional edge. Clearly, no such strategy may yield less cost than $3 + \alpha$ and, thus, cannot be an improving move. Hence, deleting her edge ea is her unique improving move, which must be her best response move. This move transforms G_4 into G_1 . \square

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